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PROGRESS REPORT.

The first Conference of our society, having proved a success to some considerable extent, there has been a strong feeling amongst our members to hold a second Conference either in Bombay or Poona. The Committee have carefully considered this matter, and now feel great pleasure in announcing to the General Body that *a Conference of our society will be held in Bombay about the beginning of January* 1919. The details will be published hereafter in the October issue of our Journal, but members desirous of attending the conference and of reading papers at the conference are requested to communicate their intentions to the undersigned, with short synopses of papers to be read.

2. The following gentlemen have been elected members of our Society

(1) *Lt. Robert Francis Mudie* I.C.S., I.A.R.O.—School of Musketry, Satara.

(2) *Mr. T. S. Krishna Murti*, B.A., L.T.—Assistant Inspector of schools, Madura.

(3) *Mr. P. T. Arumuhaswami*,—Mathematics Teacher, St. Michael's College, Mungalasthan, Batticaloa (Ceylon).

3. (i) The following book has been purchased for the Library—*Synopsis of Pure Mathematics*—G.S. Carr, M.A., published by Francis Hodgson, London 1886, £ 1—14s—0.

(ii) The following books have been received as presents to the Library :—

1. *Cours de Mecanique* (professe' a l'ecole' Polytechnique) Vol. III,—by L. Lecorm, published by Gauthier Villars, Paris, 1918.
2. *Tables Numeriqes Usuelles*—L. Zoretti, publhished by Gauthier Villars, Paris, 1917.
3. *Astrographic Catalogue*—Measures of Rectangular Coordinates and Diameters of 6 3 4 3 6 stars, images &c.—by R. J. Pocock, B.A., B.Sc.C. F.R.A.S. published by H. H. the Nizam's Government, 1918. Rs. 12 or 16s net.
4. *Madras University Calendar*—Parts I and II for 1918, and volume of Examination papers 1917.
5. *Bombay University Calendar*—Parts I and II for 1918—1919.

POONA, 8th *Aug.* 1918.

D. D. KAPADIA,
Honorary Jt. Secretary.

Properties of Equations in the Algebra of the Nodal Cubic.

BY R. VYTHYNATHASWAMY.

[The present paper purposes to give a formal development of some general function—properties in the Algebra of the Nodal Cubic, on the basis of the results already established from geometrical considerations in my previous paper on the subject.

The following is a general summary of the paper.

A. Discussion of Functional Forms.

(1) The general functional form transformable into a successive product *i.e.* a product in which the multiplier at each stage is a single element instead of being itself a product. The successive product taken as the universal standard form.

(2) Classification of forms into pure and mixed forms with respect to a variable x. Reduction of the general mixed form to three canonical types. Consequent reduction of the general equation

$$F_1(x) = F_2(x)$$

to two fundamental types,

$$P_x = a \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$P_x = a \, (x Q_x) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where P_x, Q_x are pure functions.

(3) Classification of forms into two types, the simplex and the complex. A form $f(x)$ is complex if $f(n)$ is indeterminate and simplex if $f(n) = n$ or n^1 (n, n^1 being the nodal elements) A complex form transformable into a simplex one by the rules of the Algebra.

The conditions for the simplexity of the form

$$x^2 (n p_{n-1} - - p_2 p_1)$$

are (i) the secondary indices $p_2 \, p_3 - - p_n$ are alternately even and odd or odd and even and (ii) $p_1 \, p_2$ are both odd or both even.

B. Function—Properties.

(1) The apparent degree of a function when in a complex form higher than its true degree; *e. g.* $x^2 x^2$, a complex form of the function x^3

The degree of a simplex form, in other words the true degree of any function is never a multiple of three.

(2) Classification of functions into odd and even functions. $f(x)$ is odd if $f(n) = n^1$ and even, if $f(n) = n$. The odd-ness or even-ness of $f(x)$ directly determinable if the form $f(x)$ is simplex. Whether the form $f(x)$ is simplex or complex, $f(x)$ is odd or even according as the characteristic is positive or negative.

(3) Classification of equations in complex and simplex types according as they are or are not satisfied by the nodal points. The degree of a simplex equation of the first type never a multiple of 3, the degree of a simplex equation of the second type always a multiple of 3.

(4) The problem of solving the general equation reduced to the root-extraction of a given correspondence (which is merely the problem of root extraction of a number and the geometrical problem of drawing a ray making with three rays a pencil of given cross-ratio.

The n roots of an equation of the n th degree form a periodic group of the homographic correspondence of period n whose fixed points are the nodal points.]

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§ 1. *Reciprocal points*

Two points on the Nodal Cubic connected by the relation $ab^2 = a^2$s will be said to be reciprocal. This relation is of the third degree in a and there are therefore three a's for any b. One of these three roots is however b itself. Thus there are two points reciprocal to a given point b (not including b itself). We may verify that these two pointe are $\omega_1(\omega_2 b)$ and $\omega_2(\omega_1 b)$ where $\omega_1, \omega_2, \omega_3$ are the points of inflexion. It may be noted that $\omega_1(\omega_2 x) = \omega_2(\omega_3 x) = \omega_3(\omega_1 x)$. This follows from the theorem $\omega_1(\omega_2\omega_3 x) = \omega_2 x$, proved in para (*b*), 9 '*Nodal Cubic*,' J. I. M. S. August 17.

For

$$b(\omega_1(\omega_2 b))^2 = b(\omega_1(\omega_2 b^2)) = b^2(\omega_1(\omega_2 b)).$$

Thus the two points reciprocal to b are $\omega_1(\omega_2 b)$ and $\omega_2(\omega_1 b)$.

We notice also that these two points are themselves reciprocal.

For

$$\begin{aligned} \omega_1(\omega_2(\omega_1(\omega_2 b))) &= \omega_1(b(\omega_1\omega_2)) \\ &= \omega_1(\omega_3 b) \\ &= \omega_2(\omega_1 b) \end{aligned}$$

which proves $\omega_1(\omega_2 b)$ is reciprocal to $\omega_2(\omega_1 b)$.

Note (1). If Ω denote the homographic correspondence having the nodal points for its fixed points and carrying ω_1 to ω_2 (so that $\Omega(x)=\omega_1(\omega_2 x)$), then Ω carries ω_2 to ω_3 and ω_3 to ω_1 Ω is therefore periodic, of period three. The two points reciprocal to b are then seen to be $\Omega(b)$ and $\Omega^2(b)$.

Note (2). The only self-reciprocal points are the nodal points.

Note (3). If x_1, x_2, x_3 form a mutually reciprocal set,

$$x_1^2=x_2x_3,\ x_2^2=x_3x_1,\ x_3^2=x_1x_2.$$

For
$$x_2x_3=[\omega_1(\omega_2x_1)][\omega_1(\omega_3x_1)]$$
$$=\omega_1(\omega_1x_1^2)=x_1^2.$$

Note (4). The points ω_1, ω_2, ω_3 form a mutually reciprocal set, so also do the points, $-\omega_1, -\omega_2, -\omega_3$, which are the points of contact of tangents from the inflexional points.

Note (5). $x=\Omega(x^2).\Omega^2(x)$ identically.
For $\Omega(x^2)=\omega_1(\omega_2x^2)$ and $\Omega^2(x)=\omega_1(\omega_3 x)$
$\therefore$ $\Omega(x^2).\Omega^2 x)=\omega_1^2(\omega_2\omega_3.(x.x^2))$ (by the theorem $ab.cd=ac.bd$).
$$=\omega_1(\omega_1(x.x^2)=x$$
Note (6). $[\Omega(x)]^{2n}=\Omega(x^2)$
From *Note* (5) $\Omega(x^2)=x.\Omega^2(x)$
$$=[\Omega(x)]^2(\textit{Note}\ (3)).$$

By successive applications of this, the general result follows.

§ 2. The general equation of the third degree may be reduced to one of the two forms

$$x^2=bx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$
$$ax^2=bx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

b is evidently one root of equation (1). Further if x is one nodal point n, x^2 is the other nodal point n^1 and bx is also the other nodal point n^1, so that the two other roots of equation (1) are n, n^1 the two nodal points. Equation (1) is thus degenerate and need not be considered further.

Consider now equation (2). It can be written in the form

$$x^2=S(x), \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

S denoting a homographic correspondence with the nodal points as fixed points.

The three roots of equation (2) *form a mutually reciprocal set.*

To prove this we shew that if x_1 is one root $\Omega(x_1)$ is another. We have

$$[\Omega(x_1)]^2 = \Omega(x_1^2) \; \{ \text{I, Note (6)} \}$$
$$= \Omega \, S(x_1) \; \{ \therefore x_1 \text{ is a root} \}$$
$$= S \, \Omega(x_1) \, ;$$

so that $\Omega(x_1)$ is another root. Thus the three roots of equation (2) are of the form x_1, $\Omega(x_1)$, $\Omega^2(x_1)$.

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The equation (2) thus associates each homographic correspondence with nodal points as fixed points, with a reciprocal set. The identical correspondence is associated with the inflexional triad, and the fundamental involution with the triad $-\omega_1, -\omega_2, -\omega_3$.

§ 3. *Solution of the cubic equation.*

Let Ω_x denote a homographic correspondence, with the nodal points as fixed points, carrying x to a point of inflexion. Thus Ω_x is not definite, but may be any one of the correspondences, $\left|\begin{matrix} x \\ \omega_1 \end{matrix}\right|$, $\left|\begin{matrix} x & x \\ \omega_2 & \omega_3 \end{matrix}\right|$. But it is easy to see that in any case, Ω_{x3} is definite and unique and that if $\Omega_x{}^3$ is known, x is determined as one of the set of mutually reciprocal points.

$$(\Omega_x{}^3)^{-3}\,(\omega_1),\ \Omega(\Omega_x{}^3)^{-1}\,(\omega_1)\ \Omega^2(\Omega_x{}^3)^{-3}\,(\omega_1).$$

We have the important relation

$$\Omega_x{}^3 = \left|\begin{matrix} x \\ {}^2 \\ x \end{matrix}\right|$$

For $$\Omega_x{}^3(x) = x(\omega(x(\omega(x(\omega x)))))$$
$$= x^2.$$

This property leads immediately to the solution of the equation
$$x^2 = S(x).$$

For since $x^2 = \Omega_x{}^3(x)$ identically and Ω_x and S have the same fixed points, it follows that
$$\Omega_x{}^3 = S = \text{a known correspondence}$$
whence x is determined as shewn above and the equation is solved.

§ 4. *Properties of successive products.*

Consider the successive product.
$$x_n(x_{n-1}(x_{n-2}(\ldots\ldots\ldots\ldots x_2\, x_1)))$$
We shall find it convenient to write this
$$(x_n\, x_{n-1} \ldots\ldots\ldots x_2\, x_1)$$

Let us call $x_1\ x_3\ x_5 \ldots$ the odd elements and $x_2\ x_4 \ldots$ the even elements of the product. The following results readily follow from (7) para (6) 'Nodal cubic'.

(1) All the even elements and the first odd element can be arranged among themselves in any order.

(2) All the odd elements except the first can be arranged among themselves in any order.

Theorem I. If $P_m=(p_m\ p_{m-1}\cdots p_2\ p_1)$ and $X_m=(p_1\ p_m\ p_{m-1}\cdots p_2 x)$, then $P_m X_m=x$ or $p_1^2\, x$, according as m is odd or even.

$$
\begin{aligned}
\text{For}\quad P_m X_m &= (p_m\ p_{m-1}\cdots p_2\ p_1)\times(p_1\ p_m\ p_{m-1}\cdots p_2\ x)\\
&= (p_1\ p_{m-1}\ p_{m-2}\cdots p_2\, p_1)\times(p_m\, p_m\ p_{m-1}\cdots p_2\ x)\\
&= (p_1\ p_{m-1}\cdots p_2\ p_1)\times(p_{m-1}\ p_{m-2}\cdots p_2\ x)\\
&= (p_{m-1}\ p_{m-1}\ p_{m-2}\cdots p_2\ p_1)\times p_1\ p_{m-2}\cdots p_2\ x)\\
&= (p_{m-2}\ p_{m-3}\cdots p_2 p_1)\times(p_1 p_{m-2}\cdots p_2 x)\\
&= P_{m-2}\times X_{m-2}\\
&= P_3\times X_3 \text{ if } m \text{ is odd}\\
&= (p_3 p_2 p_1)\times(p_1 p_3 p_2 x)\\
&= p_2\times p_2 x = x\,;\\
\text{and}\quad &= P_2\times X_2 \text{ if } m \text{ is even}\\
&= p_2 p_1\times(p_1 p_2 x)\\
&= p_1^2 x.
\end{aligned}
$$

Theorem II. The product of two successive products can be expressed as a successive product.

Let the two successive products be denoted by P_m, Q_n where,

$$P_m=(p_m p_{m-1}\cdots p_2 p_1),\ Q_n=(q_n q_{n-1}\cdots q_2 q_1).$$

We may suppose m to be odd. If it were even, we may write p_1 as $p_1 p_1^2$ and thus convert P_m into an odd product. We shew that

$$P_m\cdot Q_n=R \text{ where } R=(p_1 p_m\cdots p_2 q_n\cdots q_1).$$

We have

$$
\begin{aligned}
P_m R &= P_m(p_1 p_m\cdots p_2 q_n q_{n-1}\cdots q_1)\\
&= P_m(p_1 p_m\cdots p_2 Q_n)\\
&= P_m(P_m Q_n)\ (\text{Th I})\\
&= Q_n
\end{aligned}
$$

$$\therefore\quad P_m Q_n=R=\text{a successive product.}$$

Corollary. It follows by repeated application of them thereto that the most general function of a variable x, can be expressed as a successive product the elements of which are powers of x and constants.

Def. A successive product the elements of which are powers of x is said to be *a pure product.* If constants also appear, the product is said to be *mixed.*

Theorem III The most general mixed product can be reduced to one of the forms, $a\ P_x$, $a\ (bP_x)$, $x^{2m}\ (cP_x)$, where P_x is a pure product. consider first the mixed product C_x, where

$$C_x = x_1(c_1(x_2(c_2 \ldots x_n(c_n X))))$$

$x_1, x_2 \ldots X$ being pure functions of x and c's constants.

This product $= (x_1 X x_2 c_2 x_3 c_3 \ldots x_n c_n{}')$

where $c_n{}' = c_1 c_n$

$= (x_1 X x_2 c_3 c_n{}' c_3 x_4 c_4 \ldots x_{n-1} c_{n-1} x_n{}')$

where $x_n{}' = x_n x_3$

$= (x_1 X x_2 b x_4 c_4 \ldots x_{n-1} c_{n-1} x_n{}')$

where $c_2(c_n{}'(c_3 d)) = bd$

$= (x_1 X C_x{}')$

where $C_x{}'$ is of the same form as C_x but with fewer elements

Repeating this process of reduction, we would have finally

$$C_x = (y_1 y_2 y_3 \ldots y_n by)$$

or

$$C_x = (y_1 y_2 y_3 \ldots y_n aby)$$

where the y's are pure functions of x.

In the second case, C_x at once reduces to the form $a(bP_x)$ where P_x is a pure product. In the first case, C_x would reduce to one of the two forms bP_x, or $y_1\ (bP_x)$.

The latter form can be reduced to $x^{2m}(bP_x)$ by interchanging y_1 and an adequate power of x occuring in P_x expressed as a successive product.

Consider now the general mixed product. The constants occuring in it can be all shifted to the left-hand side of the product, so that if x_r, x_s be the first non-constant odd and even elements, no odd element to the right of x_r and no even element to the right of x_s are constants. The product then reduces to one of the forms C_x, $a\ C_x$, $a(bC_x)$. Taking each of the three possible forms of C_x, it is easy to verify that the product must reduce to one of the three forms mentioned in the theorem.

§ 5. Let the correspondence $\begin{vmatrix} nn^1 x \\ nn^1 x^2 \end{vmatrix}$ or as we shall write it for convenience $\begin{vmatrix} x \\ x^2 \end{vmatrix}$, be denoted by S_x. Then $\begin{vmatrix} x^2 \\ x^4 \end{vmatrix}$ would be denoted by Sx^2

Now $S_x{}^2S_{x^2}(y)=x(x^2(x(x^2x(x^2(x^4y))$
$=x(x^2(x(x^4y)))$
$=x(y(x(x^4x^2)))$
$=y.$

$\therefore \qquad S_x{}^3S_{x^2}=1$ So that $S_{x^2}=S_x{}^{-2}$.

$$\therefore \left|\begin{matrix} x \\ x^{2^n} \end{matrix}\right| = \left|\begin{matrix} x \\ x^3 \end{matrix}\right| \cdot \left|\begin{matrix} x^2 \\ x^4 \end{matrix}\right| \cdots \left|\begin{matrix} x^{2^{n-1}} \\ x^{2^n} \end{matrix}\right|$$

$$=S_x . S_{x^2} \ldots S_{x^{2^{n-1}}}$$

$$=S_x^{\frac{1-(-2)^n}{3}}$$

$$\therefore \left|\begin{matrix} x^{2^{m_1}} \\ x^{2^{m_2}} \end{matrix}\right| = S_x^{\frac{(-2)^{m_1}-(-2)^{m_2}}{3}}$$

Writing the successive product $x^{2^{m_n}}(x^{2^{m_{n-1}}}\ldots(x^{2^{m_2}}x^{2^{m_1}})))$ as $x^{2(m_n\ m_{n-1}\ldots m_1)}$, we may shew more generally that

$$L \equiv \left|\begin{matrix} x \\ x^{2(m_n\ m_{n-1}\ldots m_1)} \end{matrix}\right| = S_x^{\frac{1+(-)^n\{(-2)^{m_1}+(-2)^{m_2}-(-2)^{m_3}+-\}}{3}}$$

Proof. Assume the theorem to be true for products of n elements. We shall prove it true for products of $(n+1)$ elements. We have

$$\left|\begin{matrix} x \\ x^{2(m_{n+1},\ m_n\ldots m_1)} \end{matrix}\right| = L\times \left|\begin{matrix} x^{2(m_n\ m_{n-1}\ldots m_1)} \\ x^{2(m_{n+1}\ m_n\ldots m_1)} \end{matrix}\right|$$

$$= L\times \left|\begin{matrix} x^{2^{m_{n+1}}} \\ x_2(m'_n\ m'_{n-1}\ldots m'_2\ m'_1) \end{matrix}\right|$$

where $m'_r=m_r+1$

(This follows because $\left|\begin{matrix} p \\ qp \end{matrix}\right| = \left|\begin{matrix} q \\ p^3 \end{matrix}\right|$)

$$= L\times \left|\begin{matrix} x^{2^{m_{n+1}}} \\ x \end{matrix}\ \ \begin{matrix} \\ 1 \end{matrix}\right| \left|\begin{matrix} x \\ x^{2(m'_n\ldots m'_1)} \end{matrix}\right|$$

$$= L\times S_x^{\frac{(-2)^{m_{n+1}}-1}{3}} \times S_x^{\frac{1+(-)^{n+1}x2\times(-2)^{m_1}+(-2)^{m_2}-+\ldots)}{3}}$$

$$= S_x^{\frac{1+(-)^{n+1}((-2)^{m_1}+(-2)^{m_2}-+\ldots)}{3}}$$

The theorem is thus true for products of $(n+1)$ elements.

Since the theorem has been proved true for simple powers, the induction shews that it is universally true.

Def. The expression

$$\frac{1+(-)^{m}\left((1-2)^{p_1}+(-2)^{p_2}-(-2)^{p_3}+(-2)^{p_4}\ldots\right)}{3}$$

may be called *the characteristic* of the successive product $x^{2}{}^{(p_m p_{m-1}\ldots p_1)}$. Evidently the characteristic is an intger $+^{ve}$ or $-^{ve}$.

§ 6. A fuuction of x which $=n$ when $x=n'$ will be said to be an even function. A function which $=n'$, when $x=n$ will be saia to be an odd function.

A function which does not become indeterminate when x is made equal to a nodal point will be said to be simplex.

(a) A pure successive product is an even or odd function according as its characteristic is $-^{ve}$ or $+^{ve}$.

To prove this we observe that if $f(x)$ is an odd function $S_x[f(x)]$ is also an odd function; for

$$\begin{aligned} S_x(f(x)) &= x(x^2 f(x)) \\ &= n(n'n') \text{ (putting } x=n) \\ &= n' \end{aligned}$$

Hence $S_x(f(x))$ is an odd function. In the same way it is seen that if $f(x)$ is an even function $S_x^{-1}(f(x))$ is also an even function.

Now if P_x be the successive product and k its characteristic.

We have $\left|\begin{matrix} x \\ P_x \end{matrix}\right| = S_x^{k}$ or $P_x = S_x^{k}(x) = S_x^{k-1}(x^2)$

since x is an eveu function and x^3 an odd function it follows at once that P_x is even or odd according as k is $-^{ve}$ or $+^{ve}$

(b) To find the necessary and sufficient condition that a pure product may be simplex.

Evidently the necessary and sufficient condition is that, in the course of the product an odd function should nowhere be multiplied by an even function or an even function by an odd function.

Let the product be $x^{2}{}^{(p_m p_{m-1}\ldots p_2 p_1)}$

Firstly $x^{2^{p_1}}$ and $x^{2^{p_2}}$ should be both odd or both even functions.

Which shews that p_1 and p_2 should be both odd or both even secondly it should be even or odd according as $x^{2(p_{r-1}p_{r-2}\ldots p_2p_1)}$ is even or odd, for all values of r.

The characteristics are therefore of the same sign.

Therefore $(-)^r(-2)^{p_r}$ is of the same sign as $((-2)^{p_1}+(-2)^{p_2}\ldots$

$$\pm(-2)^{p_{r-1}})$$

This shews that $p_2p_3p_4\ldots$are alternately odd and even.

Cor. The degree of a pure simplex product cannot be a multiple of three.

The degree of the product in (*b*) above is

$$(2)^{p_1}+(2)^{p_2}+2^{(p_3)}\ldots$$

Now

$$2^{p_1}=\pm1 \pmod 3$$
$$2^{p_2}=\pm1 \pmod 3$$
$$2^{p_3}=\mp1 \pmod 3$$
$$2^{p_4}=\pm1 \pmod 3$$

The upper signs being taken if p_1 is even, the lower if p_1 is odd. (These equations follow by applying the conditions that the product may be simplex).

Adding up, we see that the degree can never be a multiple of three.

§ 7. *The General Equation.*

We are now in a position to discuss the general equation

$$F_1(x)=F_2(x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

F_1, and F_2 can be reduced to successive products [IV] and further reduced to one of the three forms aP_x, $a(bP_x)$, x^{2m} (cP_x) where P_x is a pure product. It is easy to verify that the equation (1) can in all cases be reduced to one of the two types

$$P_x=a \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$
$$P_x=a(bQ_x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where P_x and Q_x are pure products.

An equation is said to be simplex, when the nodal points are not roots of it. Evidently the equation of the first type is simplex if P_x is simplex and the equation of the second type is simplex if P_x and Q_x are not both odd or both even. We prove from this the interesting result:—*The degree of a simplex equation of the first type is never a multiple of 3; the degree of a simplex equation of the second type is always a multiple of 3.*

The first part follows from § 6, Cor.

To prove the second part, we refer back to § 6, Cor.

The sign of the characteristic of the product $x^{2}\,(p_m p_{m-1} \ldots p_1)$ is $(-)^{p_1+m}$. The degree of the product is of the form) $3q+(-)^{m-1})$ if p_1 is even and of the form $(3q+(-)^{m})$ if p_1 is odd ; i.e. of the form $(3q-(-)^{p_1+m})$ whether p_1 is odd or even. Hence the degree of a simplex product is of the form $(3q \mp 1)$ according as its characteristic is $+^{ve}$ or $-^{ve}$. Applying this result to the equation of the second type, we see that the degree of the one of the functions P_x, Q_x is of the form $39+1$, and of the other, of the form $3q-1$, so that the degree of the equation itself is a multiple of 3.

Solution of the General Equation.

(i) *The equation of the first type* $P_x = b$.

Let k be the characteristic of P_x, so that

$$\overset{k}{S_x} = \left| \begin{matrix} x \\ P_x \end{matrix} \right| = \left| \begin{matrix} x \\ b \end{matrix} \right| = \left| \begin{matrix} x \\ \omega_1 \end{matrix} \right| \left| \begin{matrix} b \\ \omega_1 \end{matrix} \right|^{-1}$$

$$\text{Now } S_x = \left| \begin{matrix} x \\ \omega_1 \end{matrix} \right|^{3}$$

$$\therefore \quad \left| \begin{matrix} x \\ \omega_1 \end{matrix} \right|^{3k-1} = \left| \begin{matrix} b \\ \omega_1 \end{matrix} \right|^{-1} = \text{a known correspondence}$$

This determines x. We see incidentally that the degree of the equation supposed simplex is $3k-1$.

(ii) *The equation of the second type* $P_x = a(bQ_x, = S(Q_x)$

If k, k' are the characteristics of P_x, Q_x, we have

$$S = \left| \begin{matrix} Q_x \\ P_x \end{matrix} \right| = \underset{= S_x}{\left| \begin{matrix} x \\ P_x \end{matrix} \right|^{k-k'}} \underset{= \Omega_x}{\left| \begin{matrix} x \\ Q_x \end{matrix} \right|^{-1}}^{3(k-k')}$$

which determines Ω_x^{3} and $\therefore x$.

We see that the degree of the equation is the numerical value of $3(k-k')$.

§ 8. The last paragraph shews that the solution of any equation is reduced to the ertraction of roots and to the geometrical problem of drawing a ray making with three other rays, a pencil of given cross ratio. It also shews that if α is one root of an equation of the nth

degree, all the roots are given by the series, α, $\Omega_n(\alpha)$, $\Omega^2_n(\alpha)$......... $\Omega_n^{n-1}(\alpha)$, Ω_n being a primitive n^{th} root of the identical correspondence with the nodal points as fixed points.. This is merely another aspect of the fact that in this algebra every equation involves only one parameter. This somewhat strange result is shewn by the theory of linear systems of points to be necessitated by the symmetrical relation between the nodal points and the linear system formed by the roots of all equations of a particular degree. Now there are only two ways in which *two* points can be related in absolute symmetry to a linear system, viz. (1) the system may be a complete one and the two points may form the indeterminate pair, in which case the order of the system must be three; This happens in the case of the nodal points and the complete system of collinear triads; or (2) the system must be a focal system of the second degree involving a single parameter with the two points as foci. The linear system of roots of equations of a particular degree belongs to this latter category.

SHORT NOTES.

Some Fundamental Expansions.*

§ 1. Consider $\dfrac{\cos\lambda x}{\sin\rho x}$ as the limit, when m becomes infinite, of

$$f(m)=i\frac{\left(1+\dfrac{i\lambda x}{m}\right)^m+\left(1-\dfrac{i\lambda x}{m}\right)^m}{\left(1+\dfrac{i\rho x}{m}\right)^m-\left(1-\dfrac{i\rho x}{m}\right)^m},$$

and suppose that m reaches infinity through odd positive integral values. Decomposing $f(m)$ into partial fractions, the factors of the denominator are obtained by solving

$$\left(1+\frac{i\rho x}{m}\right)^m-\left(1-\frac{i\rho x}{m}\right)^m=0.$$

This gives

$$\frac{1+\dfrac{i\rho x}{m}}{1-\dfrac{i\rho x}{m}}=1^{\frac{1}{m}}=e^{\frac{2ik\pi}{m}} \quad \text{[where } k \text{ is zero or any integer]}$$

$$\therefore \quad i\rho x=\frac{e^{\frac{ik\pi}{m}}-e^{-\frac{ik\pi}{m}}}{e^{\frac{ik\pi}{m}}+e^{-\frac{ik\pi}{m}}}=i\tan\frac{k\pi}{m}, \text{ i.e. } x=\frac{m}{\rho}\tan\frac{k\pi}{m}.$$

Denoting this value of x by x_k, the partial fractions are

$$\frac{1}{\rho x}+\Sigma\frac{\delta_k}{x-x_k},$$

where by the usual rules

$$\delta_k=\left[\left(1+\frac{i\lambda x_k}{m}\right)^m+\left(1-\frac{i\lambda x_k}{m}\right)^m\right]\div$$
$$\left[\left(1+\frac{i\rho x_k}{m}\right)^{m-1}+\left(1-\frac{i\rho x_k}{m}\right)^{m-1}\right]$$

and k takes the values, $-\dfrac{m-1}{2}, -\dfrac{m-3}{2}, \ldots -1, +1, \ldots \dfrac{m-1}{2}$,

Also since $x_k=-x_{-k}$ and $\delta_k=\delta_{-k}$,

* The four fundamental expansions contained in this Note can be obtained by a method similar to that given in Todhunter's *Integral Calculus*, § 29. The method here given is, perhaps, more rigid.

$$\therefore \qquad f(m)=\frac{1}{\rho x}+\sum\frac{2x\delta_k}{x^2-x_k^2},$$

the summation extending from $k=1$ to $k=\frac{1}{2}(m-1)$.

Now $(1+i\lambda x_k/m)^m+(1-i\lambda x_k/m)^m$

$$=\left(1+i\lambda\tan\frac{k\pi}{m}\Big/\rho\right)^m+\left(1-i\lambda\tan\frac{k\pi}{m}\Big/\rho\right)^m$$

$$=(1+i\tan\theta_k)^m+(1-i\tan\theta_k)^m, \text{ putting } \tan\theta_k=\frac{\lambda}{\rho}\tan\frac{k\pi}{m},$$

$$=2\sec^m\theta_k\cos m\theta_k;$$

and $(1+i\rho x_k/m)^{m-1}+(1-i\rho x_k/m)^{m-1}$

$$=\left(1+i\tan\frac{k\pi}{m}\right)^{m-1}+\left(1-i\tan\frac{k\pi}{m}\right)^{m-1}$$

$$=2\sec^{m-1}\frac{k\pi}{m}\cos(m-1)\frac{k\pi}{m}$$

$$=2(-)^k\sec^{m-1}\frac{k\pi}{m}\cos\frac{k\pi}{m}.$$

Hence $\delta_k=\dfrac{(-1)^k}{\rho}\left(\dfrac{\cos k\pi/m}{\cos\theta_k}\right)\dfrac{\cos m\theta_k}{\cos^2\frac{k\pi}{m}}.$

For any particular large value of m, put

$$f_1(m)=\frac{1}{\rho x}+\sum_{k=1}^{k=p}\frac{2x\delta_k}{x^2-x_k^2}, \text{ and } f_2(m)=\sum\frac{2x\delta_k}{x^2-x_k^2};$$

the second summation extending from $k=p+1$ to $k=\frac{1}{2}(m-1)$; then $f(m)=f_1(m)+f_2(m)$. Now let m become infinite; then since $f(m)$ and $f_1(m)$ have each a definite limit, $f_2(m)$ has also a definite limit;

also $\qquad \underset{m=\infty}{\mathrm{Lt}}\tan\dfrac{k\pi}{m}=\dfrac{k\pi}{m}$; and $\underset{m=\infty}{\mathrm{Lt}}\delta_k=\dfrac{(-1)^k}{\rho}\cos\dfrac{\lambda k\pi}{\rho}$;

and since $\cos\dfrac{\lambda k\pi}{\rho}\not>1$, we may suppose p to be taken so great that the numerical value of $f_2(m)$ is less than

$$\sum\frac{2|x|}{|\rho|(k^2\pi^2/\rho^2-x^2)}$$

This last is a convergent series, and therefore when p is taken sufficiently large, $f_2(m)$ can be made less than a quantity as small as we please. Proceeding to the limit when p tends to m or ∞, we get the fundamental form

$$\frac{\cos\lambda x}{\sin\rho x}=\frac{1}{\rho x}+\frac{1}{\rho}\underset{m=\infty}{\mathrm{Lt}}\sum_{k=1}^{k=m}\frac{(-1)^k\cos\frac{\lambda k\pi}{\rho}}{(x^2-k^2\pi^2/\rho^2)}2x. \quad \dots \quad \dots \quad (A)$$

In particular if $\rho=1=\lambda$, we have

$$\cot x=\frac{1}{x}+\underset{m=\infty}{\mathrm{Lt}}\sum_{k=1}^{k=m}\frac{(-1)^k\cos k\pi}{(x^2-k^2\pi^2)}2x=\frac{1}{x}+\sum_{k=1}^{k=\infty}\frac{2x}{x^2-k^2\pi^2},$$

as is well known; and by putting $\lambda=0$, $\rho=1$, we get

$$\operatorname{cosec} x=\frac{1}{x}+\sum_{k=1}^{k=\infty}\frac{(-1)^k}{x^2-k^2\pi^2}2x.$$

We can similarly find out the expansions of the remaining three forms, namely

$$\frac{\sin\lambda x}{\sin\rho x}, \frac{\cos\lambda x}{\cos\rho x}, \frac{\sin\lambda x}{\cos\rho x}$$

by reasoning as for the first fundamental form: the following convergent expansions will be obtained:—

$$\frac{\sin\lambda x}{\sin\rho x}=\sum_{k=1}^{k=\infty}\frac{(-1)^k 2\pi k\sin\frac{\lambda k\pi}{\rho}}{(\rho^2x^2-k^2\pi^2)}, [\lambda<\rho],$$

$$=\underset{m=\infty}{\mathrm{Lt}}\sum_{k=-m}^{k=+m}\frac{(-1)^k\sin\frac{\lambda k\pi}{\rho}}{(\rho x-k\pi)}; \quad \dots \quad \dots \quad (B)$$

$$\frac{\cos\lambda x}{\cos\rho x}=\sum_{k=1}^{k=8}\frac{(-1)\cos\frac{\lambda}{\rho}(2k-1)\frac{\pi}{2}}{x^2-(2k-1)\frac{\pi^2}{4\rho^2}}\cdot\frac{(2k-1)\pi}{\rho^2}, [\lambda<\rho]$$

$$=\sum_{k=1}^{k=\infty}\frac{(-1)^k\cos\frac{\lambda}{\rho}(2k-1)\frac{\pi}{2}}{\rho^2x^2-(2k-1)^2\frac{\pi^2}{4}}\cdot(2k-1)\pi; \quad \dots \quad (C)$$

and

$$\frac{\sin\lambda x}{\cos\rho x}=\sum_{k=1}^{k=\infty}\frac{(-1)^k\rho\sin\frac{\lambda}{\rho}(2k-1)\frac{\pi}{2}}{\rho^2x^2-(2k-1)^2\frac{\pi^2}{4}}2x, [\lambda<\rho]$$

$$=\underset{m=\infty}{\mathrm{Lt}}\sum_{k=-m}^{k=+m}\frac{(-1)^k\sin(2k-1)\frac{\lambda\pi}{2\rho}}{\rho x-(2k-1)\frac{\pi}{2}}\rho. \dots \quad \dots \quad (D)$$

In (B) and (D), the value $k=0$ should not be included.

These results may also be obtained from (A) by observing

$$\frac{\sin\lambda x}{\sin x}=\frac{1}{\sin\lambda\pi}\left\{\cos\lambda\pi\frac{\cos\lambda x}{\sin x}+\frac{\cos\lambda(x+\pi)}{\sin(x+\pi)}\right\}$$

and using similar trigonometrical identities.

Taking the first fundamental form and changing λ to $i\lambda$, we get

$$\frac{\cosh\lambda x}{\sin\rho x}=\frac{1}{\rho x}+\frac{1}{\rho}\sum_{k=1}^{k=\infty}\frac{(-1)^k\cosh\frac{k\lambda\pi}{\rho}}{(x^2-k^2\pi^2/\rho^2)}2x. \quad \dots \quad \dots \quad (A')$$

Further changing ρ to $i\rho$, we get

$$\frac{\cos\lambda x}{\sinh\rho x}=\frac{1}{\rho x}+\frac{1}{\rho}\sum_{k=1}^{k=\infty}\frac{(-1)^k\cosh\frac{\lambda k\pi}{\rho}}{(x^2+k^2\pi^2/\rho^2)}2x. \quad \dots \quad \dots \quad (A'')$$

Lastly changing λ to $i\lambda$ and ρ to $i\rho$, we get

$$\frac{\cosh\lambda x}{\sinh\rho x}=\frac{1}{\rho x}+\frac{1}{\rho}\sum_{k=1}^{k=\infty}\frac{(-1)^k\cos\frac{k\lambda\pi}{\rho}}{(x^2+k^2\pi^2/\rho^2)}2x. \quad \dots \quad \dots \quad (A''')$$

Taking the second fundamental form (B) and making similar changes, we shall have—

$$\frac{\sinh\lambda x}{\sin\rho x}=\sum_{k=1}^{k=\infty}\frac{(-1)^k 2\pi k\sinh\frac{\lambda k\pi}{\rho}}{\rho^2x^2-k^2\pi^2}, \quad \dots \quad \dots \quad (B')$$

$$\frac{\sin\lambda x}{\sinh\rho x}=\sum_{k=1}^{k=\infty}\frac{(-1)^{k-1}2\pi k\sinh\frac{\lambda k\pi}{\rho}}{\rho^2x^2+k^2\pi^2}, \quad \dots \quad (B'')$$

and

$$\frac{\sinh\lambda x}{\sinh\rho x}=\sum_{k=1}^{k=\infty}\frac{(-1)^{k-1}2\pi k\sin\frac{\lambda k\pi}{\rho}}{\rho^2x^2+k^2\pi^2}. \quad \dots \quad \dots \quad (B''')$$

Similarly from the third fundamental form (C) we get

$$\frac{\cosh\lambda x}{\cos\rho x}=\sum_{k=1}^{k=\infty}\frac{(-1)^k\cosh\frac{\lambda}{\rho}(2k-1)\frac{\pi}{2}}{x^2-(2k-1)^2\frac{\pi}{4\rho^2}}(2k-1)\frac{\pi}{\rho^2}, \quad \dots \quad (C')$$

and from the fourth fundamental form (D)

$$\frac{\sinh\lambda x}{\cos\rho x}=\sum_{k=1}^{k=\infty}\frac{(-1)^k\rho\sinh\frac{\lambda}{\rho}(2k-1)\frac{\pi}{2}}{\rho^2x^2-(2k-1)^2\frac{\pi^2}{4}}\,2x. \quad \dots \quad \dots \quad (\text{D}')$$

§ 2. A large number of results may be obtained from these forms by various analytical methods.

For example :—

If $z=\sin\rho x, \frac{\sin\lambda x}{\sin\rho x}=\frac{1}{z}\sin\left(\frac{\lambda}{\rho}\sin^{-1}z\right)$; hence by the ordinary expansion we have

$$\frac{\sin\lambda x}{\sin\rho x}=\frac{\lambda}{\rho}\left\{1+\frac{1}{3\,!}\left(1-\frac{\lambda^2}{\rho^2}\right)\sin^2\rho x\right.$$

$$\left.+\frac{1}{5\,!}\left(1-\frac{\lambda^2}{\rho^2}\right)\left(3^2-\frac{\lambda^2}{\rho^2}\right)\sin^4\rho x+\dots\right\}$$

$$=\frac{2}{\pi}\sin\frac{\lambda\pi}{\rho}\left(1-\rho^2x^2/\pi^2\right)^{-1}-\frac{4}{2^2\pi}\sin\frac{2\lambda\pi}{\rho}$$

$$(1-\rho^2x^2/2^2\pi^2)^{-1}+\dots\text{by result (B)};$$

from which, by equating coefficents of different powers of x, we derive the following series

$$\sin\frac{\lambda\pi}{\rho}-\frac{1}{2}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3}\sin\frac{3\lambda\pi}{\rho}-\quad\dots\quad=\frac{\pi}{2}\cdot\frac{\lambda}{\rho}\quad \dots \quad \dots \quad (1)$$

$$\frac{1}{1^3}\sin\frac{\lambda\pi}{\rho}-\frac{1}{2^3}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3^3}\sin\frac{3\lambda\pi}{\rho}-\dots=\frac{\pi^3}{12}\left(\frac{\lambda}{\rho}-\frac{\lambda^3}{\rho^3}\right)\quad \dots \quad (2)$$

$$\frac{1}{1^5}\sin\frac{\lambda\pi}{\rho}-\frac{1}{2^5}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3^5}\sin\frac{3\lambda\pi}{\rho}$$

$$=\frac{\pi^5}{720}\frac{\lambda}{\rho}\left(1-\frac{\lambda^2}{\rho^2}\right)\left(1-\frac{3\lambda^2}{\rho^2}\right)\quad \dots \quad (3)$$

$$\frac{1}{1^7}\sin\frac{\lambda\pi}{\rho}-\frac{1}{2^7}\sin\frac{2\lambda\pi}{\rho}+\frac{1}{3^7}\sin\frac{3\lambda\pi}{\rho}-$$

$$=\frac{\pi^7}{2}\frac{\lambda}{\rho}\left(1-\frac{\lambda^2}{\rho^2}\right)\left\{\frac{2}{45}-\frac{1}{240}\left(3^2-\frac{\lambda^2}{\rho^2}\right)\right.$$

$$\left.+\frac{1}{7\,!}\left(3^2-\frac{\lambda^2}{\rho^2}\right)\left(5^2-\frac{\lambda^2}{\rho^2}\right)\right\}\quad \dots \quad \dots \quad (4)$$

The series (1), (2), (3), (4), are all convergent.

[See : Chrystal, Chap XXVI, §9]

M. K. Kewalramani.

An Interesting Experiment.

[The following interesting experiment described in December 29, 1917 issue of "Tit-Bits" was successfully performed by the present writer, and is reported here in view of its scientific interest. The details of the experiment can be varied, at pleasure, of course.]

Take a good-sized bowl, fill it nearly with water, and place it upon the floor of a room which is not exposed to shaking or jarring from the street traffic. Sprinkle over the surface of the water a coating of lycopodium powder, which can be obtained at any hospital or chemists' shop. Then upon the surface of this coating of powder make with powdered charcoal a straight black line, say an inch or two in length. Having made this little mark with the charcoal powder on the surface of the contents of the bowl, lay down upon the floor close to the bowl a stick (or some other straight object) so that it will be exactly parallel to the mark. (If the line happens to be parallel to a crack in the floor, or to any stationary object in the room, this will serve as well.)

Leave the bowl undisturbed for a few hours, and then observe the position of the black mark, with reference to the object to which it was parallel. It will be found to have moved in the direction opposite to that of the movement of the earth on its axis The earth is simply rotating and has carried the water and everything else in the bowl round with it, but the powder on the surface has been left behind a little. The line will always be found to have moved from east to west, which is a perfectly good proof that everything else has moved the other way.

K. B. M.

An interesting Cubic Connected with an Ellipse.

Basset in his *Treatise on Cubic and Quartic Curves* has discussed the different forms of the cubics obtained by inverting a conic with respect to its vertex. (vide : §§ 125, 146-7). The following interesting case has however been overlooked by him :

Let an ellipse be referred to the extremity of its minor axis as origin. Then its equation is written

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2y}{b}$$

Inverting with respect to the origin, we have the circular cubic

$$2y\frac{(x^2+y^2)}{b}=k^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right),$$

whose form is shown below.

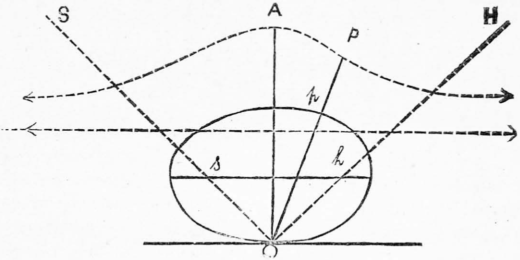

Fig. 1.

If S, H be the inverses of the foci s, h of the ellipse. and P, p be any two corresponding points on the cubic and the ellipse, we readily obtain,

$$\frac{\mathrm{SP}}{\mathrm{OS}}+\frac{\mathrm{HP}}{\mathrm{OH}}=\frac{\mathrm{OP}}{k^2}(sp+hp)$$

$$=2a.\frac{\mathrm{OP}}{k^2}.$$

But $\quad os=oh=a,\ \mathrm{OS}\cdot os=k^2=\mathrm{OH}.oh.$

$\therefore \quad \mathrm{OH}=\mathrm{OS}=k^2/_a.$

Hence $\quad \mathrm{SP}+\mathrm{HP}=2\cdot\mathrm{OP}\,,$

In other words, the tri-polar equation of the cubic is

$$r_1+r_2=2r_3\ .$$

M. T. Naraniengar.

Degenerate Bicircular Quartics.

The equation of two inverse circles (the origin being the centre of inversion) is

$$(r^2+2cx+\delta^2\lambda)(r^2+2cx/\lambda+\delta^2/\lambda)=0.$$

Comparing this with equation (12) on p. 136 of Basset; *Cubic and Quartic Curves*, we infer that a bicircular quartic will degenerate in two inverse circles, if

$$4f = 2c\left(1+\frac{1}{\lambda}\right),\ g=0,$$

and

$$4f^2+2\delta^2-4a^2=\frac{4c^2}{\lambda}+\delta^2\left(\lambda+\frac{1}{\lambda}\right),$$

$$2\delta^2-4b^2=\delta^2\left(\lambda+\frac{1}{\lambda}\right).$$

Thus

$$f^2 = c^2/\lambda + (a^2-b^2)$$

$$=\frac{4f^2\lambda}{(1+\lambda)^2}+(a^2-b^2);$$

also

$$\delta^2(\lambda+\frac{1}{\lambda}+2)=4(\delta^2-b^2).$$

$$\therefore\quad f^2\left(1-\frac{\delta^2}{\delta^2-b^2}\right)=a^2-b^2.$$

$$\therefore\quad f^2=\frac{(a^2-b^2)(b^2-\delta^2)}{b^2}$$

$$\therefore\quad f=\pm\sqrt{\left\{\frac{(a^2-b^2)(b^2-\delta^2)}{b^2}\right\}}.\qquad \ldots \qquad \ldots \qquad (1)$$

In other words, if the centre of inversion is the point $(f, 0)$ where f is determined by (1), the bicircular quartic reduces to two inverse circles.

Examples :

(i) Let the centre of inversion be the foot of the normal at $P(\phi)$; then

$$f=\frac{(a^2-b^2)}{a}\cos\phi;$$

$$\therefore\quad \left(1-\frac{b^2}{a^2}\right)\cos\phi=\left(1-\frac{\delta^2}{b}\right)\text{ by (1).}$$

$$\therefore\quad \delta^2=\frac{b^2}{a^2}(a^2 sin^2\phi+b^2 cos^2\phi)=\frac{b^2.CD^2}{a^2},$$

$$=PG^2.$$

Hence, we have the following result :

If from G a perpendicular GT be drawn on any tangent and a point R taken on this perpendicular such that $GT^2-RT^2=PG^2$, then the locus of R is either of two circles.

(ii) Suppose $\phi=0$ in the above. Then $f=(a^2-b^2)/a$, and $\delta=b^2/a$.

For points outside the limits $\pm\ (a^2-b^2)/a$, the value of δ is real up to $\sqrt{(a^2-b^2)}$, and beyond that δ becomes imaginary, being the length of the imaginary normal from the point to the ellipse,

M. T. Naraniengar.

Astronomical Notes.

The New Star in Aquila.

The new star in Aquila appeared very suddenly on June 8 last when it was about the same magnitude as Altair, for about two days it increased in magnitude till it became nearly equal to Vega and then began to fade steadily. At first the star was white in colour, but about five days after its first appearance it became distinctly yellow, and a few days later became red. The spectrum about this stage showed a velocity of approach of about one-thousand miles per second, similar to that observed in Nova Persei.

The Nova appeared in the dark rift in the Milky way about one hour west of Altair, its approximate position being R.A. $18^h\ 44^m\ 47^s$ Decl. $0^\circ\ 31'$ N.

This is the fourth Nova which has been discovered in Aquila. In 389 A.D. a bright Nova appeared close to Altair, but vanished after three weeks. In 1899 and again in 1905 photographic discoveries of Nova in Aquila were made at Harvard College Observatory. The position of the present Nova however differs greatly from any of the other three so that identity is not possible.

An interesting Problem.

The sixth and seventh satellites of Jupiter revolve about their primary at approximately the same distance and consequently in approximately the same period. The same is true of the eighth and ninth satellites, the sidereal periods of which are 739 days and 745 days respectively, a difference of less than one per cent. Now it appears that the perijories of the sixth and seventh satellites are separated by approximately 180°, while the same is found to be true of the eighth and ninth satellites and the question arises, is this purely accidental or is there any dynamical reason ? for example, is this arrangement more stable than any other ?

It may be mentioned that whereas the motion of the sixth and seventh satellites is direct that of the eighth and ninth is retrograde and it was shown a few years ago that at a considerable distance from the primary a retrograde satellite is more stable than a direct satellite having the same mean distance.

Relativity.

A great deal continues to be published on the subject of Relativity, both with respect to Einstein's recent theory and Sir Oliver Lodge's attempt to explain the motion of the perihelion of Mercury, theories, both of which have been previously referred to in this Journal. I cannot give all the references but the following paper may be found of interest by those interested in the subject.

Planetary motion in space—Time of any constant curvature according to the generalised principle of relativity ; by L. Silberstein. *Mon. Not. R. A. S.* 1918 March, p. 363.

R. J. Pocock.

SOLUTIONS.

Question 612.

(N. Ganapathi Subba Aiyar):—A series of ellipses have a common focus (S), pass through a given point (P) and have their major axes of the same length. Find their envelope.

Additional Solution by N. Durairajan.

Lemma: If two curves touch, their first positive pedals touch. Therefore if a number of curves have a common envelope the pedals of the curves have also a common envelope which is the the pedal of the original envelope.

In the present case, the pedals of the ellipses are circles of constant radii and also the circles on SP as diameter touches these circles. The envelope of these circles clearly consists of two circles of radii $2a+2r$, where a denotes the semi-major axis and r denotes SP.

Hence the envelope of the ellipses is the negative pedal of these two circles and consists of two conics having a common focus S.

Question 625.

(V. V. S. Narayan):—Show that the lines joining the vertices of a triangle to the corresponding vertices of its first Brocard triangle concur in a point Q whose trilinear co-ordinates are inversely proportional to those of the pole of $\Omega\ \Omega'$ with respect to the Brocard circle,

Solution by K. B. Madhava.

Let K be the symmedian point, and EF′, FD′, DE′ be the antiparallels though it cutting the circle on SK (S is the circumcentre) as diameter at the points A′, B′, C′ respectively. Then A′B′C′ is said to be the first Brocard triangle and the circle as the Brocard circle.

Its equation in trilinears will be seen to be (by the method of Askwith *Coordinate Geometry*: p. 416. Ex. 4,).

$$\alpha^2+\beta^2+\gamma^2-\frac{a^2}{bc}\beta\gamma-\frac{b^2}{ca}\gamma\alpha-\frac{c^2}{ab}\alpha\beta=0.$$

Since the first Lemoine circle goes through EF′ and is concentric with the Brocard circle through K, A′; KE = A′F′; *i.e.* AA′ and AK

cut BC in points equidistant from its midpoint. Therefore by Ceva's theorem AK′, BB′, CC′ concur and their point of intersection (Q) is the isotomic conjugate of K, *i.e.* its trilinear coordinates are

$$\left(\frac{1}{a^3}, \frac{1}{b^3}, \frac{1}{c^3}\right).$$

Moreover since the normal coordinates of

$$\Omega \text{ are } \frac{c}{b}, \frac{a}{c}, \frac{b}{a} \text{ and of } \Omega^1 \ \frac{b}{c}, \frac{c}{a}, \frac{a}{b} \qquad \ldots \qquad \ldots \qquad (2)$$

the equation of $\Omega\,\Omega^1$ is

$$\alpha\left(a^3-\frac{b^2c^2}{a}\right)+\beta\left(b^3-\frac{a^2c^2}{b}\right)+\gamma\left(c^3-\frac{b^2a^2}{c}\right)=0 \qquad \ldots \qquad (3)$$

and its pole with respect to the circle (1) is (a^3, b^3, c^3). Hence the result stated.

Question 734.

(J. C. Swaminarayan) :—Solve the differential equation

$$(y^2-b^2)\frac{dy}{dx}+xy=\sqrt{b^2x^2+a^2y^2-a^2b^2}.$$

Solution by N. Durairajan.

Write $x=a\cos(\alpha\cdot v.)$

$y=b\sin(\alpha\cdot v.)$

Substituting in the equation and reducing we get

$$d\alpha+dv.\frac{\sqrt{v^2-1}}{v}=0.$$

The variables are separated and the equation may be considered as solved.

Question 753.

(S. Ramanujan) :—If

$$\phi(x)=\tfrac{1}{2}\log 2\pi x-x+\int_1^x\frac{[t]}{t}dt$$

where $[t]$ denotes the greatest integer in t, show that

$$\overline{\lim_{x\to\infty}}\ x\phi(x)=\frac{1}{24} \text{ and } \underline{\lim_{x\to\infty}}\ x\phi(x)=-\frac{1}{12}.$$

Solution (1) *by N. Durairajan* ; (2) *by* K. B. *Madhava.*

(1) We have

$$\phi(x)=\frac{1}{2}\log 2\pi x-x+\int_1^x\frac{[t]}{t}dt.$$

Expressing the integral as I, we have

$$I=\int_1^2 \frac{[t]}{t}\,dt+\int_2^3 \frac{[t]}{t}\,dt+\dots+\int_{[x]}^{x} \frac{[t]}{t}dt.$$

$$=1\log\frac{2}{1}+2\log\frac{3}{2}+3\log\frac{4}{3}+\dots+[x]\log\frac{x}{[x]}.$$

$$=\log\left[\frac{2}{1}\cdot\frac{3^2}{2^2}\cdot\frac{4^3}{3^3}\cdots\frac{x^{(x)}}{[x]^{(x)}}\right]$$

$$=\log\frac{x^{(x)}}{[x]\,!}$$

We have

$$\phi(x)=\log\frac{e^{-x}\sqrt{2\pi x}.\;x^{(x)}}{[x]\,!}$$

Now let $x=[x]+\delta=n+\delta$, and keeping δ constant make $n\to\infty$. Now

$$\lim_{n\to\infty}(x\phi(x))=\lim_{n\to\infty}(n+\delta)\log\frac{e^{-n-\delta}\sqrt{2\pi(n+\delta)}(n+\delta)^n}{n\,!}$$

δ being constant.

$$=\lim_{n\to\infty}(n+\delta)\log\frac{e^{-n}\sqrt{2\pi n}.n^n\left(1+\frac{\delta}{n}\right)^{n+\frac{1}{2}}e^{-\delta}}{n\,!}$$

$$=\lim_{n\to\infty}(n+\delta)\log e^{-\frac{1}{12n}-\theta}$$

$$+(n+\delta)\log\left\{e^{-\delta}\left(1+\frac{\delta}{n}\right)^{n+\frac{1}{2}}\right\}$$

by Stirling's theorem, where $\theta=0\left(\frac{1}{n^2}\right)$:

$$=(n+\delta)\left(-\frac{1}{12}\;-\theta\right)+(n+\delta)\left(\frac{\delta-\delta^2}{2n}+\frac{4\delta^3-3\delta^2}{12n^2}+\dots\right)$$

$$=-\frac{1}{12}+\frac{1}{2}(\delta-\delta^2)-\frac{\delta}{12n}(1-6\delta+2\delta^2)+\dots$$

In other words, the limit depends on δ. The maximum and minimum values of $\delta-\delta^2$ are when $\delta=\frac{1}{2}$ and 0, and equal to $\frac{1}{4}$ and 0 respectively. Hence $\lim x\phi(x)$ oscillates between the limits

$$-\frac{1}{12}+\frac{1}{2}\frac{1}{4}\quad\text{and}\quad-\frac{1}{12}+0\ \textit{ie}\text{ between }\frac{1}{24}\text{ and }-\frac{1}{12},$$

(2) Differentiating we have

$$\frac{d\phi}{dx}=\frac{1}{2x}-1+\frac{n}{x},$$

which vanishes when $x=n+\frac{1}{2}$.

Also $\frac{d^2\phi}{dx^2}=-\frac{1+2n}{2x^2}=$ negative when $n\to\infty$

Hence we infer that

$$\overline{\lim_{x\to\infty}}\, x\phi(x)=\lim_{n\to\infty}\,(n+\tfrac{1}{2})\phi(n+\tfrac{1}{2}).$$

(Vide Solution to Q 740, page 220., Vol. VIII)

Question 783.

(S. Ramanujan) :—If $x=y^n-y^{n-1}$, and

$$J_n=\int_0^1 \log y\,\frac{dx}{x},$$

show that

(i) $J_0=\frac{\pi^2}{6}$; $J_{\frac{1}{2}}=\frac{\pi^2}{10}$; $J_1=\frac{\pi^2}{12}$; $J_2=\frac{\pi^2}{15}$.

(ii) $J_n+J_{\frac{1}{n}}=\frac{\pi^2}{6}$.

Solution by N. Durai Rajan and 'Zero'

(I) If $x=y^n-y^{n-1}$, it is obvious that for positive values of x, $y>1$ when positive. Also, y cannot be negative, since the question involves log y. Thus, the values of y corresponding to the limits (0 , 1) of x are (1 , α), where

$$\alpha^n-\alpha^{n-1}-1=0 \quad \text{and } \alpha>1.$$

Again, if $x=y^{\frac{1}{n}}-y^{\frac{1}{n}-1}$, the similar limits for y are $(1,\beta)$, where

$$\beta^{\frac{1}{n}}-\beta^{\frac{1}{n}-1}-1=0$$

which may be written (after multiplication by $\beta^{1-\frac{1}{n}}$)

$$\beta-\beta^{1-\frac{1}{n}}-1=0$$

proving that $\beta=\alpha^n$, and $\frac{1}{\alpha}+\frac{1}{\beta}=1$. ... ... (1)

Now

$$J_n=\int_0^1 \log y \, d\,(\log x),$$

$$=[\log x \log y]-\int_1^{\alpha} \log x \, d\,(\log y),$$

$$J_n=-\int_1^{\alpha}\left\{ n \log y+\log\left(1-\frac{1}{y}\right)\right\}\frac{dy}{y},$$

since the value of $(\log x \log y)$ between the limiting values of x and y is easily seen to be zero.

Hence

$$J_n=-\left\{\frac{n}{2}(\log y)^2+\int\log\left(1-\frac{1}{y}\right)\frac{dy}{y}\right\}_1^{\alpha}.$$

$$=-\tfrac{1}{2}(\log\alpha\log\beta)-\int_1^{\alpha}\log\left(1-\frac{1}{y}\right)\frac{dy}{y}. \qquad \ldots \quad (2)$$

Similarly

$$J_{\frac{1}{n}}=-\frac{1}{2}(\log\alpha\log\beta)-\int_1^{\beta}\log\left(1-\frac{1}{y}\right)\frac{dy}{y}. \qquad \ldots \qquad \ldots \quad (3)$$

To prove that $J_n+J_{\frac{1}{n}}$ is constant, we differentiate (2) and (3) with respect to α and β and remember that

$$\frac{d\alpha}{\alpha^2}+\frac{d\beta}{\beta^2}=0 \text{ by virtue of (1.)}$$

$$\frac{1}{\alpha}+\frac{1}{\beta}=1.$$

Thus

$$\frac{dJ_n}{d\alpha}=-\frac{1}{2}\left\{\frac{\log\beta}{\alpha}+\frac{\log\alpha}{\beta}\frac{d\beta}{d\alpha}\right\}-\frac{1}{\alpha}\log\left(1-\frac{1}{\alpha}\right)$$

$$=-\frac{1}{2}\frac{\log\beta}{\alpha}+\frac{\beta\log\alpha}{2\alpha^2}+\frac{\log\beta}{\alpha}$$

$$=\frac{(\alpha\log\beta+\beta\log\alpha)}{2\alpha^2};$$

and

$$\frac{dJ_{\frac{1}{n}}}{d\beta}=\frac{(\alpha\log\beta+\beta\log\alpha)}{2\beta^2}.$$

$$\therefore \qquad \frac{d\left(J_n+J_{\frac{1}{n}}\right)}{d\alpha}=0, \text{ since } \frac{d\alpha}{\alpha^2}+\frac{d\beta}{\beta^2}=0.$$

Hence $J_n + J_{\frac{1}{n}}$ is a constant for all values of n.

In particular $J_1 + J_1 = 2\left(\frac{\pi^2}{12}\right) = \frac{\pi^2}{6}$, by the first part. Finally, therefore, we have

$$J_n + J_{\frac{1}{n}} = \frac{\pi}{6},$$

II. Since $x = y^n - y^{n-1} = y^n\left(1 - \frac{1}{y}\right)$ is transformed, by the substitution $\frac{1}{\eta} = 1 - \frac{1}{y}$, into

$$x^{\frac{1}{n}} = \left(1 - \frac{1}{\eta}\right)\ \eta^{\frac{1}{n}},$$

we see that

$$J_{\frac{1}{n}} = \int_0^1 \frac{d\xi}{\xi} \log \eta, \text{ where } \xi = x^{\frac{1}{n}}, \ \frac{1}{\eta} = 1 - \frac{1}{y}.$$

Thus $J_{\frac{1}{n}} = \int_{\infty}^1 \frac{1}{n}\frac{dx}{x} \log\left(1 - \frac{1}{y}\right).$

$$\therefore\ J_n + J_{\frac{1}{n}} = \int_0^1 \frac{dx}{x}\log y + \int_{\infty}^1 \frac{1}{n}\frac{dx}{x}\log\left(1 - \frac{1}{y}\right)$$

$$= \int_0^{\infty} \frac{dx}{x}\log y + \int_{\infty}^1 \left\{\frac{dx}{x}\log y + \frac{dx}{nx}\log\left(1 - \frac{1}{y}\right)\right\}$$

$$= \int_1^{\infty}\left[n\frac{dy}{y} + d\log\left(1 - \frac{1}{y}\right)\right]\log y + \int_{\infty}^1 \frac{\log x}{n}\, d\,(\log x)$$

$$= \left[\frac{n}{2}(\log y)^2 - \frac{1}{2n}(\log x)^2\right]_1^{\infty} + \int_1^{\infty} \log y\ d\log\left(1 - \frac{1}{y}\right)$$

$$= \left[\log y\left(1 - \frac{1}{y}\right) - \frac{1}{2n}\left(\log 1 - \frac{1}{y}\right)^2 - \log y\left(1 - \frac{1}{y}\right)\right]_1^{\infty}$$

$$- \int_1^{\infty} \log\left(1 - \frac{1}{y}\right)\frac{dy}{y}$$

$$= \frac{\pi^2}{6},$$

since the expression within the square brackets is zero.

Question 800.

(S. MALHARI RAO) :—Shew that the sum of all fractions which may be represented by a recurring decimal of the form $(.\dot{a}\,\dot{b}\,c\,d)$ is 50, provided $a+c=b+d=9$.

Further Remarks by H. Br.

The error in printing the question on p. 199 of Vol. IX was fortunate; and so was the failure of the attempt on p. 251 of the same volume to correct this error. For, the solutions on p. 326 of the last issue of the Journal show that the second dot might be placed on a or on b without altering the proposition. On looking more closely into the matter I find that the second dot might also have been placed on c or nowhere at all. The moral credit for the following generalization thus belongs to the Printer's Devil.

If $(.ab\ldots\dot{g}h\ldots\dot{n})$ is a recurring fraction (pure or mixed) its value is equal to

$$Aa+Bb+\ldots+Nn$$

where the quantities A, B,..,N are determined by the positions of the corresponding digits, but are independent of the value of those digits. As a general rule A, B,...N vary with the variations in the form of the recurring fraction, but for all such fractions

$$A+B+\ldots+N=\frac{1}{9}$$

as is clear from the fact that the expression on the left is the value of the continued fraction when all the digits are equal to unity.

The number of the digits and the position of the dots being fixed, let us obtain all possible recurring fractions as under :—

(i) The digits in k assigned positions are given any value we please, independently of each other ;

(ii) Each digit in any of the other positions is dependent on a digit in the first group in such a way that there is a complete 1-1 correspondence between the ten possible values of the two. Subject to this limitation, the law of dependence is arbitrary and not necessarily the same for any two pairs, and each of the independent digits may have any number (from zero upwards) of digits depending upon it.

The proposition is that the sum of all fractions is $\frac{1}{2}.10^{k}$

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Proof :—The sum is

$$A\Sigma a + B\Sigma b + \ldots + N\Sigma n$$

where the sammation extends over all admissible values of each digit.

But the 1–1 correspondence ensures that the summation for each of the two connected digits is the same. Clearly the summation for all the independent digits is the same, *i.e.*

$$10^{k-1}.(0+1+2+\ldots+9) = 10^{k-1} \times 45 = \frac{9}{2}.\ 10^{k}.$$

Therefore the sum of all fractions is

$$\frac{9}{2}.10^{k}.(A+B+\ldots+N) = \frac{1}{2}10^{k}.$$

Whatever k be, the average value of the fractions is $\frac{1}{2}$ as it should be because the average value of *all* possible fractions is $\frac{1}{2}$, and we have taken a fair sample of *all* fractions.

Question 925.

(S. R. RANGANATHAN) :—If P, N, N_0, N_1, are the corresponding points on a curve, its evolute, the evolute of the evolute and the evolute of the α—evolute respectively, show that N_1 describes a circle on $N_0 N$ as diameter, as α varies from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

Solution by R. Srinivasan, M. A., M. R. A. S.

Let P, P^1, P^{11} be consecutive points on the curve, N, N^1 on the evolute ;

$$\text{then } N\hat{P}N_1 = N\hat{P^1}N_1 = \alpha.$$

$$\therefore\ N\hat{N_1}P_1 = N\hat{P}P^1 = 90^\circ \text{ ultimately.}$$

$$\therefore\ N_1\hat{N}N_0 = 90^\circ - N_1\hat{N}P^1 = \alpha.$$

Similarly $N_1\hat{N^1}N_0 = \alpha$ ultimately.

$\therefore$ $N_1 N N^1 N_0$ is cyclic.

$$\therefore\ N\hat{N_1}N_0 = 180^\circ - N\hat{N^1}N_0 = 90^\circ \text{ ultimately}$$

$\therefore$ N_1 lies on the circle on $N N_0$ as diameter.

As α (*ie* $N_1\hat{N}N_0$) varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, N_1 will describe the circle completely.

Question 866

(H. R. KAPADIA) :—Prove that

$$\int_{\infty} x^{-2n} \sum_{1}^{n}(A_r \cos a_r x)\, dx = \frac{\pi \cos n\pi}{(2n-1)} \sum_{1}^{n} (A_r a_r^{2n-1}).$$

Solution by K. B. Madhava, V. D. Gokhale, Sadanand, M, K. Kewalramani, S. V. Venkatachela Iyer, and K. R. Rama Iyer.

Since the integrand is even, it is equal to twice the value of

$$\int_{0}^{\infty} x^{-2n} \sum_{1}^{n} (A_r \cos a_r x)\, dx.$$

Integrating this by parts we are successively led to

$$\int_{0}^{\infty} \frac{1}{2n-1} \sum_{1}^{n} A_r a_r (\sin a_r x) \frac{dx}{x^{2n-1}}$$

and
$$\int_{0}^{\infty} \frac{-1}{(2n-1)(2n-2)(2n-3)} \sum_{1}^{n} A_r a_r^{3} \sin a_r x) \frac{dx}{x^{2n-3}}$$

and so on ;

assuming $\Sigma A_r = 0$; $\Sigma A_r a_r^2 = 0$ up to

$$\int_{0}^{\infty} \frac{(-)^n}{(2n-1)!} \Sigma(A_r a_r^{2n-1}) \frac{\sin x}{x} dx$$

$$= \frac{\pi}{2} \cdot \frac{\cos n\pi}{(2n-1)!} \Sigma(A_r a_r^{2n-1}).$$

Hence the result.

Note : It is easy to see that the allied integral

$$\int_{0}^{\infty} x^{2n-1} \sum_{1}^{n}(A_r \cos a_r x)\, dx$$

leads us to
$$\frac{\cos n\pi}{2n!} \int_{0}^{\infty} \Sigma(A_r a_r^{2n} \cos x) \frac{dx}{x}$$

with the conditions $\Sigma A_r = 0$; $\Sigma A_r a_r^2 = 0 \ldots \Sigma A_r a_r^{2n} = 0$

which gives finally
$$\frac{\cos n\pi}{2n!} \Sigma(A_r a_r^{2n} \log a_r)$$

by Frullani's theorem. (cf. Bromwich, § 170).

Question 867.

(C. Krishnamachari) :—If the velocity of the Earth is instantaneously altered in the ratio of $\sqrt{2}:1$, when it is at the extremity of the minor axis of its elliptic orbit, show that the new orbit subsequently described is a parabola whose latus-rectum is twice of the elliptic orbit.

Solution by S. V. Venkatachala Iyer, M. K. Kewalramani, Sadanand, S. Muthukrishnan and S. R. Ranganathan.

The velocity at any point of the orbit is given by

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right).$$

$\therefore$ At the given instant the velocity is given by

$$v^2=\mu\left\{\frac{2}{a}-\frac{1}{a}\right\}=\frac{\mu}{a}.$$

$\therefore$ v_1^2, the square of the new velocity $=\frac{2\mu}{a}$

$\therefore$ the subsequent motion is given by

$$v^2=\frac{2\mu}{r}$$

which shows that the subsequent path is a parabola.

The semi-latus rectum in the first case is given by $l=\frac{h^2}{\mu}=\frac{v^2p^2}{\mu}$

and in the second by $l_1=\frac{p}{\mu}$. Since p, the perpendicular from the origin to the tangent to the path is the same in both cases

$$l:l_1=v^2:v_1^2=\frac{\mu}{a}:\frac{2\mu}{a}=1:2.$$

Question 888.

(Hemraj) :—ABCD is a tetrahedron, the edges DA, DB, DC are a,b,c the angles BDC, CDA, ADB are λ,μ,ν; and 2 X is the sum of the areas of the four faces; and 2 Y the sum of the areas of the four faces less twice the area of the face opposite D; and V the volume. Show that

$$\frac{4X^2Y^2}{9V^2}\Delta\operatorname{cosec}^2\sigma=\Sigma a^2\sin^2(\sigma-\lambda)-\Sigma bc(\cos\lambda-\cos\mu-\nu)$$

where $$2\sigma=\lambda+\mu+\nu,$$

and $$\Delta=1-\Sigma\cos^2\lambda+2\cos\lambda\cos\mu\cos\nu.$$

Solution by H. Br, K. Appukuttan Erady, H. R. Kapadia, K. B. Madhava and S. V. Venkatachala Iyer.

Let α, β, γ, δ be twice the area of the faces. Then
$\alpha = bc \sin\lambda$, and $\beta = ca \sin\mu$, $\gamma = ab \sin\nu$.

By considering δ to be the resultant of forces α, β, γ, acting per pendicular to these faces we get

$$\delta^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\Sigma\beta\gamma \cos(\beta\gamma)$$
$$= \alpha^2 + \beta^2\gamma^2\, 2\Sigma a^2bc \sin\mu \sin\nu \cos(\beta\gamma)$$
$$= \alpha^2 + \beta^2 + \gamma^2 - 2abc\Sigma a \cos\lambda - (\cos\mu\cos\nu)]$$
$$= (\alpha+\beta+\gamma)^2 - 2abc\Sigma a\,(\cos\lambda - \cos\overline{\mu+\nu})$$
$$= (\alpha+\beta+\gamma)^2 - 4abc\Sigma a \sin\sigma \sin(\sigma-\lambda)$$

$\therefore$ $$4XY = \tfrac{1}{4}[(\alpha+\beta+\gamma)^2 - \delta^2] = abc \sin\sigma\Sigma a \sin(\sigma-\lambda)$$

or $$\frac{16X^2Y^2}{a^2b^2c^2}\operatorname{cosec}^2\sigma = \Sigma a^2 \sin^2(\sigma-\lambda) + 2\,\Sigma\, bc \sin(\sigma-\mu)\,\delta \sin(\sigma-\nu)$$
$$= \Sigma a^3 \sin^2(\sigma-\lambda) - bc\,(\cos\lambda - \cos\overline{\mu-\nu})$$

From this the given result follows with the help of the well-known formula

$$36V^2 = a^2b^2c^2(1-\cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu)$$

Question 915.

(CHAS. SALDANHA) :—Given $(a+b)$, $(a+c)$ and A, give a geometrical construction for the ΔABC

Solution (1) *by R. Srinivasan,* (2) *by K. J. Sanjana and K. B. Madhava* (3) *by T. Krishna Rao and several others.*

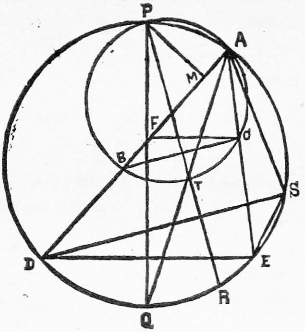
(1) Let ABC be a triangle and I_1 the excentre opposite A. Produce AB, AC to D and E such that BD = BC = CE.

Then $\Delta\, DBI \equiv \Delta\, CBI \equiv \Delta\, CEI$.

$$\therefore\ D\hat{I_1}E = 3B\hat{I_1}C = 3\left(90^\circ - \frac{A}{2}\right).$$

Also $I_1 D = I_1 C$, ; $I\,E = I\,B$. :—

Hence the following construction.

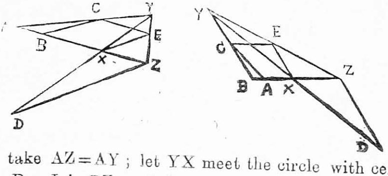


Construct a triangle ADE, such that $AD=a+c$, $AE=a+b$ and $DAE=A$. On DE describe an arc of a circle containing $3\left(90^\circ-\frac{A}{2}\right)$. Let the bisector of the angle A meet this arc in I, with centre I, and radii I D and I E describe circles to cut AE, AD, in CB. Then ABC is the triangle required.

The proof is evident.

(2) More generally let two lengths AX, AY be given inclined at a known angle A ; to draw a transversal BC across them so that

$$XB=BC=CY.$$



On AX take $AZ=AY$; let YX meet the circle with centre Z and radius ZA, in D. Join DZ, and draw XE parallel to it to meet YZ; draw EC prallel to AX to meet AY, and CB parallel to EX to meet AX or AX produced.

Then $\frac{CE}{EY}=\frac{AZ}{ZY}=\frac{ZD}{ZY}=\frac{EX}{EY}$, so that $CE=EX$.

∴ CEXB is a rhombus; and $CE=CY$, as $\angle CYE=\angle AZE=\angle CEY$

Thus XB = BC = CY and in the first figuee ABC is the triangle required by the question. In the second figure where $AX < \frac{1}{2}AY$, this triangle does not geometrically exist and $AB + BX = AX$ *algebraically.*

(3) Construct a triangle ADE such that $AD = a + c$, $AE = \overline{a} + b$ and $\angle DAE = A$. Draw the circum-circle of this triangle and PQ the diameter of the circle perpendicular to DE. Join AQ. Set off a chord QR equal to the difference of the given lengths. Join PR cutting AQ in T. Draw the circle PAT cutting AD, AE in BC respectively. Join BC. Then ABC is the required triangle.

Draw DS parallel BC to meet circle ADE in S and CF parallel to DE to meet AD in F. Join AS, SE. Draw PM perp. to AD.

$$\therefore \; PR \perp DS \text{ and } PQ \perp DE.$$
$$\angle QPR = \angle EDS$$
$$i.e. \text{ chord } ES = QR = c - b.$$

Since P is the mid point of segment DAE and PM perpendicular to AD

$\therefore$ $\quad AM = \frac{1}{2}(AD - AE)$

Similarly $\quad AM = \frac{1}{2}(AB - AC)$

i.e. $\quad AD - AE = AB - AC$ or $BD = CE$.

In the triangles FBC and AES,

the triangles $\quad B = ADS = AES$

and $\quad FCB = SDE = EAS$

hence they are similar.

i.e. $$\frac{BC}{FB} = \frac{AE}{ES} = \frac{a+b}{c-b}.$$

Again since FC is parallel to DE

$$\frac{DF}{DA} = \frac{EC}{EA} = \frac{DF - EC}{DA - EA} = \frac{BF}{c-b}.$$

or $$\frac{EC}{BF} = \frac{EA}{c-b} = \frac{a+b}{c-b}.$$

$\therefore$ $\quad BC = EC = DB.$

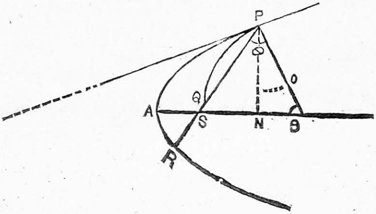
Question 916.

(N. P. Pandya) :—A circle touches a given parabola at P internally and is such that the focal distance of P is divided harmonically by the circle and the parabola. Find the locus of the centre of the circle.

Solution by Hemraj.

Let the equation of the parabola be $y^2=4ax$ and P $(at^2, 2at)$. If O, a point (x, y) on the normal at P, be the centre and P O $=r$, then

$$x=at^2+r\cos\theta,\ y=2at-r\sin\theta \qquad \dots \qquad \dots \qquad (1)$$



where $\qquad$ SPO $=\theta$ *i.e.*, $\tan\theta=t$.

Since $\qquad (PQSR)=-1,\ \dfrac{2}{PS}=\dfrac{1}{PQ}+\dfrac{1}{PR}\dots\dots \qquad \dots \qquad (2)$

But $\qquad PS=a+at^2,\ PQ=2r\cos\theta=2(x-at^2)$ from (1)

$$PR=PS+SR=2a+a\left(t^2+\frac{1}{t^2}\right)$$

$\therefore$ On substitution in (2) we have

$$3at^4+2(3a-x)t^2+a-4x=0 \qquad \dots \qquad \dots \qquad (3)$$

From (1) $\qquad at^3+(2a-x)t-y=0 \qquad \dots \qquad \dots \qquad (4)$

Eliminating t between (3) O (4) we have the locus of O

$$(4x-a)(x^2-6ax+a^2)^2+y^2(2x^3-78ax^2+90a^2x-18a^3-27ay^2)=0.$$

Question 918.

(C. KRISHNAMACHARY) :—Show that if a and b are both less or greater than unity, and $a<b$, $u_n=\dfrac{1-a^n}{1-b^n}$ is a decreasing monotonic sequence. Deduce the inequalities,

(1) $\quad \dfrac{1-b^n}{1-b}.(1-a)>(1-a^n)>\dfrac{1-b^n}{1-b}\left(\dfrac{a}{b}\right)^{n-1}(1-a), \qquad$ if $0<a<1$

(2) $\quad \dfrac{b^n-1}{b-1}(a-1)>(a^n-1)>\dfrac{b^n-1}{b-1}\left(\dfrac{a}{b}\right)^{n-1}(a-1) \qquad$ if $a>1$.

Examine the inequalities for fractional values of n.

Solution by K. B. Madhava and S. R. Ranganathan.

In general, if a sequence $\left(\frac{a_n}{b_n}\right)$ is monotonic, the sequence $\left(\frac{\Sigma a_n}{\Sigma b_n}\right)$ is also monotouic and is varying in the same sense.

For,

$$\frac{\Sigma a_n}{\Sigma b_n} > \frac{\Sigma a_{n+1}}{\Sigma b_{n+1}}$$

if

$$\frac{\Sigma a_n}{\Sigma b_n} > \frac{a_{n+1}}{b_{n+1}}$$

i.e. if $\sum_{r=1}^{n} \left(a_r b_{n+1} - b_r a_{n+1}\right) > 0$ which is true, because each of the term is positive in cosequence of the sequence $\left(\frac{a_n}{b_n}\right)$ being non-increasing

Again if the sequence is non-increasing

$$\frac{a_1}{b_1} > \frac{\Sigma a_n}{\Sigma b_n} \text{ for } \sum_{r=2}^{n} \left(a_1 b_r - a_2 b_r\right) > 0.$$

Hence combining

$$\frac{a_1}{b_1} > \frac{\Sigma a_n}{\Sigma b_n} > \frac{a_{n+1}}{b_{n+1}}. \qquad \dots \qquad \dots \qquad \text{(A)}$$

Take as $a_n = a^{n-1}(1-a)$ or $a^{n-1}(a-1)$ if $o < a < 1$ or $a > 1$

and $b_n = b^{n-1}(1-b)$ or $b^{n-1}(b-1)$ if $o < b < 1$ or $b > 1$

and then $a_n = \frac{\Sigma a_n}{\Sigma b_n}$ is a non-increasing monotonic sequence, and the inequalities (A) become (1) and (2) in the problem. (The second involves the condition $a < b$).

The inequalities may be verified for fractional values of n in the usual manner, by multiplying with the denominator, and thus proving it for an integral value of n. The reader may be referred to the examples in p. 21 of Bromwich : *Theory of Infinite Series* for interesting exercises of this type.

Question 920.

(H. R. KAPADIA) :—Prove that an integral of

$$\frac{dx}{X^{\frac{1}{2}}} = \frac{dy}{Y^{\frac{1}{2}}}$$

is

$$\left(\frac{X^{\frac{1}{2}} + Y^{\frac{1}{2}}}{y - x}\right)^2 = c + e(x+y) + f(x+y)^2$$

where $X = a + bx + cx^2 + ex^3 + fx^4$, and $Y = a + by + cy^2 + ey^3 + fy^4$

Solution by K. J. Sanjana, R. J. Pocock and K. B. Madhava and others.

This is Ex. 2 on p. 241 of Forsyth (Second Edn.).

If we assume $p=x+y$;

$$\frac{dx}{dt}=\frac{X^{\frac{1}{2}}}{y-x};\ \frac{dy}{dt}=\frac{Y^{\frac{1}{2}}}{y-x};$$

the solution proceeds exactly as in Forsyth p. 240 with one or two slight changes which are obvious. We obtain finally

$$\left(\frac{dp}{dt}\right)^2=ep+fp^2+c$$

whence substituting for $\frac{dp}{dt}$ we get

$$\left[\frac{X^{\frac{1}{2}}+Y^{\frac{1}{2}}}{y-x}\right]^2=C+e(x+y)+f(x+y)^2$$

[*Vide* also, Boole: *Differential Equations.*, Ch. VI; Roberts: *Integral Calculus* § § 87, 88; Dixon: *Elliptic Functions*, § 40.]

Question 923.

(M. K. Kewalramani) :—Supposing $\phi(x)$ vanishes when $x=0$, but neither it nor its differential coefficients vanish for any other value of x prove the following identity :—

$$x\phi(x)-\frac{x^2}{2\,!}\phi'(x)+\frac{x^3}{3\,!}\phi''(x)-\frac{x^4}{4\,!}\phi'''(x)+\dots$$
$$\equiv A_1x\phi(x)+A_3x^3\phi''(x)+A_5x^5\phi''''(x)+\dots$$

where

$$A_{2n+1}=2\frac{(2^{2n}-1)}{(2n)\,!}B_{2n-1},\ (B_{2n-1}\text{ being the }n^{th}\text{ Bernoullian.})$$

Also, show that

$$\frac{\pi}{2}A_1-\frac{\pi^3}{2^3}A_3+\frac{\pi^5}{2^5}A_5-\dots=1$$

Remarks by Sadanand.

The identity is not true as it stands. We are required to prove that

$$\left(1-e^{xD}\right)\equiv\tan\left({}^{1}xD\right)$$

where D stands for the operative symbol $\frac{d}{dx}$, which is impossible.

The series

$$\frac{\pi}{2}A_1-\frac{\pi^3}{2^8}A_3+\dots=-i\tan\left(i\frac{\pi}{4}\right)\neq 1;$$

but the series

$$\frac{\pi}{2}A_1+\frac{\pi^8}{2^3}A_8+\dots=\tan\frac{\pi}{4}=1.$$

QUESTIONS FOR SOLUTION.

[N.B.—Question 965 by Mr. A. C. L. Wilkinson which appeared in June 1918 should be numberedas 965 A.]

969. (H. Br.) :—From the vertex D of a tetrahedron ABCD straight lines DL, DM, DN, are drawn parallel and proportional to the perpendiculars from a point P onthe faces BCD, CAD, ABD, respectively. Q is the isogonal conjugate of P with respect to ABCD. Show that PD passes through the circumcentre of DLMN, and QD is perpendicular to LMN. Show further that if DL, DM, DN. are twice the lengths of the three perpendiculars, but are drawn in the opposite direction in each case, P is the circumcentre of DLMN.

970, (S. R. Ranganathan) :—Show that

$$x^n \prod_{r=1}^{\infty}\left(1+\frac{x^{2n}}{r^{2n}}\right)=\frac{1}{2^{n/2}\pi^n}\prod_{r=1}^{n-1}(\cosh 2\pi\alpha_r x-\cos 2\pi\beta_r x)$$

when n is even,

$$=\frac{1}{2^{\frac{1}{2}(n-r)}\pi^n}\quad \sinh \pi x\prod_{1}^{n-2}(\cosh 2\pi\alpha_r x-\cos 2\pi\beta_r x)$$

when n is odd,

where $\alpha_r=\sin\frac{r\pi}{2n}$, and $\beta_r=\cos\frac{r\pi}{2n}$, r being an odd integer.

971. (A. A. Krishnaswami Aiyangar M.A., L.T.) :—A rigid spherical shell of radius a contains a mass M of gas in which the pressure is K times the density, and the gas is repelled from a fixed external point O (distant c from the centre) with a force per unit of mass equal to K times the distance. Prove that the esultant pressure of the gas on the shell is

$$\frac{KM}{c}\cdot\frac{5c^2-a^2}{5c^2+a^2}.$$ (Besant and Ramsay) *Hydrostatecs.*

972. (K. B. Madhava) ;—Solve.

$$\wp(x)+\wp(y)+\wp(z)=B.$$

$$0=[\wp(y)-\wp(x)][\wp'(y)+\wp'(z)]+[\wp(y)-\wp(z)][\wp'(y)+\wp'(x)]$$

$$0=[\wp(x)-\wp(z)][\wp'(x)+\wp'(y)]+[\wp(x)-\wp(y)][\wp'(x)+\wp'(z)]$$

where $\wp$ is the Weiertrassian Elliptic function and $\wp'$, its derivate. [This is needed in the solution of Lame's equation when $n=3$;

$$\frac{1}{w}\frac{w^2d}{dr^2}=12\,\wp\,(z)+5B.].$$

973. (A. C. L. Wilkinson) :—Prove that

$$\text{Dsn}\,(u_1+u_2+u_3)=\begin{vmatrix} s_1, & s_1^1, & s_1^2 \\ s_2, & s_2^1, & s_2^2 \\ s_3, & s_3^1, & s_3^2 \end{vmatrix};\ \text{Dcn}\,(u_1+u_2+u_3)=-\begin{vmatrix} c_1, & c_1^1, & c_1^2 \\ c_2, & c_2^1, & c_2^2 \\ c_3, & c_3^1, & c_3^2 \end{vmatrix};$$

$$k^4\,\mathrm{Ddn}(u_1+u_2+u_3) = -\begin{vmatrix} d_1, & d_1^1, & d_1^2 \\ d_2, & d_2^1, & d_2^2 \\ d_3, & d_3^1, & d_3^2 \end{vmatrix};$$

where $D = \begin{vmatrix} 1, & s_1^2, & s_1c_1d_1 \\ 1, & s_2^2, & s_2c_2d_2 \\ 1, & s_3^2, & s_3c_3d_3 \end{vmatrix}$ and $s_1 = \mathrm{sn}u_1$, $s_1' = \dfrac{d}{du_1}(\mathrm{sn}u_1) = \mathrm{cn}u_1\,\mathrm{dn}u_1$ etc.

Prove that

$$\mathrm{E}(u_1)+\mathrm{E}(u_2)+\mathrm{E}(u_3)-2\mathrm{E}(x)$$
$$= \frac{k^2d_1d_2d_3d_4}{k'^2+d_1d_2d_3d_4}\left(\frac{s_1c_1}{d_1}+\frac{s_2c_2}{d_2}+\frac{s_3c_3}{d_3}+\frac{s_4c_4}{d_4}\right)$$
$$-\frac{2k^2\mathrm{sn}x\,\mathrm{cn}x\,\mathrm{dn}(x-u_3-u_4)\,\mathrm{dn}\,(x-u_2-u_4)\,\mathrm{dn}\,(x-u_2-u_3)}{k'^2+\mathrm{dn}\,x\,\mathrm{dn}\,(x-u_3-u_4)\,\mathrm{dn}\,(x-u_2-u_4)\,\mathrm{dn}\,(x-u_2-u_3)}$$

where $x=\frac{1}{2}(u_1+u_2+u_3+u_4)$ and s_1, c_1, d_1 are written as usual for $\mathrm{sn}u_1$, $\mathrm{cn}u_1$, $\mathrm{dn}u_1$.

974. (A. C. L. WILKINSON) :—If two confocal conics are such that triangles ABC can be inscribed in the one which are circumscribed to the other and if D, E, F are the points of contact, prove that (1) the area of the triangle DEF is proportional to the product of the diameters conjugate to D, E, F : (2) the ratio of the areas of the triangles ABC. DEF is constant : (3) the loci of the points of concurrence of the normals at A, B, C and D, E, F are ellipses (4) the line joining the points of concurrence of the normals at A, B, C and D, E, F is a normal to an ellipse : (5) if from the points of concurrence the fourth normals be drawn to the two confocals the line joining their feet is also normal to an ellipse.

(Suggested by Question 947 which shows that in all triangles ABC the ratio R : r is constant.)

975. (Professor K. J. SANJANA):—A uniform right pyramidal solid on a square base is divided into two halves by a plane through the vertex and one diagonal of the base, and the two parts are hinged together at the vertex. If the solid floats in a liquid which is such that the plane of flotation is a square whose side bears to the side of the base the ratio r:1, prove that the parts will not separate if $\sec\theta \gtrless r^{\frac{1}{2}}$ when the vertex is within the fluid, and if $\operatorname{cosec}\theta \gtrless (3r^3-r^2-r)^{\frac{1}{2}}$ when the vertex is without,—θ being the dihedral angle between the planes of the base and a slant side of the pyramid.

976. (K. J. SANJANA, M. A.) :—Find the probability that the roots of (i) the biquadratic $ax^4+bx^2+c=0$, (ii) the cubic $ax^3+bx+c=0$, may be all real, where the numbers a, b, c are taken at random.

977 (M. K. KEWALRAMANI) :—In the series

$$\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{19}\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \infty$$

if the terms are deranged in such a manner that to every 6561 positive terms correspond 256 negative terms, the terms in each group having their original order, show that the sum of the new series is

$$\frac{\sqrt{2}}{8}\left[\pi-\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]+\log\frac{3}{2}$$

978. M. K. KEWALRAMANI) :—If two circles are such that quadrilaterals can be inscribed in the one which are circumscribed to the other, prove that the ratio of the sum of the reciprocals of the diagonals of any two quadrilaterals is constant.

[Suggested by Q. 963]

979. (F. H. V. GULASEKHARAM) :—If $u_1, u_2, u_3, u_4, u_5, u_6$ are variable arguments connected by the relations $u_2-u_1=u_3-u_2=u_4-u_3=u_5-u_4=u_6-s=\frac{K}{3}$

and if d_r stands for $dn\ u_r$, shew that

(1) $(d_1d_2+d_2d_3+d_3d_4+d_4d_5+d_5d_6+d_6d_1)$ is constant.

(2) $(d_1+d_4)(d_2+d_5)+(d_2+d_5)(d_3+d_6)+d_3+d_6)(d_1+d_4)$is constant.

In each case determine the constant in terms of the modulus k and the sn, cn, dn, functions of $\frac{K}{3}$.

980. (F. H. V. GULASEKHARAM) :—A triangle $\triangle$ is inscribed in a given conic $S=0$ and circumscribed to a given confocal conic $S_1=0$. A triangle $\triangle_1$ is inscribed in the conic $S_1=0$ with its sides parallel to those of $\triangle$. Shew that

(1) the circum-radius of the triangle $\triangle_1$=the in radius of the triangle $\triangle$;

(2) the locus of the circum-centre of the triangle $\triangle_1$ for all positions of $\triangle$ is another fixed confocal conic.

981. (R. VYTHYNATHASWAMY). $A_1\ A_2 \ldots A_n$ are n unequal real numbers If L_{pq}...and...G_{pq} denote respectively the least and greatest of the numbers A_p, A_q,......, prove that

$$L_{123-n}=\Sigma G_1-\Sigma G_{12}+\Sigma G_{123}-\ldots$$
$$G_{123-n}=\Sigma L_1-\Sigma L_{12}+\Sigma L_{123}-$$

Use these results to express the G. C. M (L. C. M.) of n positive integers as a function of the integers and their mutual L.C M's (G.C.M's).

982. (R. VYTHYNATHASWAMY) :—If I and SIS are both involutions, prove that S must be an involution unless I contains the fixed points of S

983. (R. VYTHYNATHASWAMY) :—If Ω and S Ω are both periodic and of the same period, shew that their fixed points make with the fixed points of S ranges of equal cross ratio.